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Definition 3.8 **Gaussian Random Variable**

X is a Gaussian (μ, σ) random variable if the PDF of X is

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2},$$

where the parameter μ can be any real number and the parameter $\sigma > 0$.

Theorem 3.12 *If X is a Gaussian (μ, σ) random variable,*

$$E[X] = \mu \qquad \text{Var}[X] = \sigma^2.$$

تمرین: ثابت کنید اگر $X \sim N(\mu, \sigma^2)$ باشد. $E(X)=\mu$ است. (μ و σ^2 پارامترهای توزیع)

جواب: $E(X) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} x e^{-(x-\mu)^2/2\sigma^2} dx \quad ; \quad x = (x-\mu)+\mu$

$$= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} (x-\mu) e^{-(x-\mu)^2/2\sigma^2} dx + \mu \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-(x-\mu)^2/2\sigma^2} dx$$

$$E(x) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} y e^{-y^2/2\sigma^2} dy + \mu \int_{-\infty}^{\infty} f(x) dx \qquad y :\triangleq x-\mu$$

که در آن $f(x)$ چگالی نرمال است. بنابر تقارن انتگرال اول برابر صفر است پس:

$$E(X) = \mu \int_{-\infty}^{\infty} f(y) dy = \mu$$

Theorem 3.13 *If X is Gaussian (μ, σ), $Y = aX + b$ is Gaussian $(a\mu + b, a\sigma)$.*

Definition 3.9 **Standard Normal Random Variable**

The ***standard normal random variable*** *Z is the Gaussian $(0, 1)$ random variable.*

Definition 3.10 ***Standard Normal CDF***

The CDF of the standard normal random variable Z is

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-u^2/2}\, du.$$

Theorem 3.14 *If X is a Gaussian (μ, σ) random variable, the CDF of X is*

$$F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right).$$

The probability that X is in the interval $(a, b]$ is

$$P[a < X \leq b] = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right).$$

$$z = \frac{x-\mu}{\sigma} \tag{3.54}$$

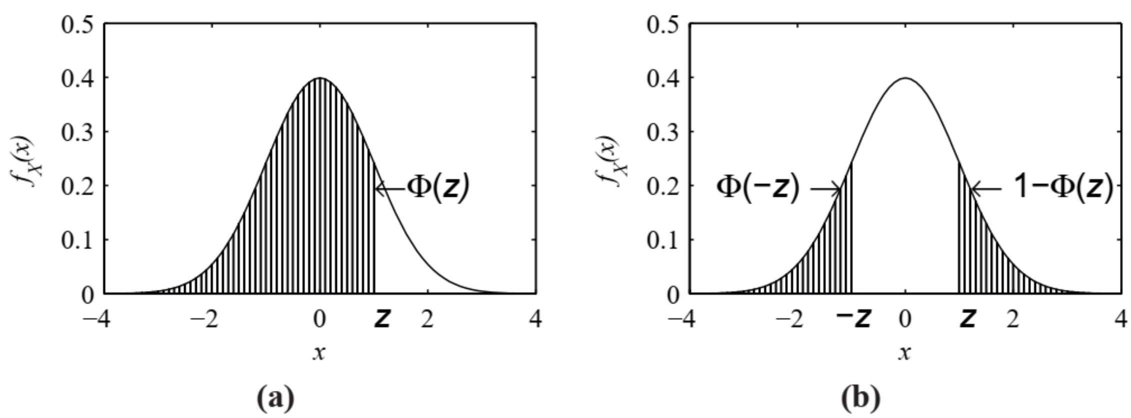

Figure 3.6 Symmetry properties of Gaussian(0,1) PDF.

Theorem 3.15

$$\Phi(-z) = 1 - \Phi(z).$$

Example 3.15 Suppose your score on a test is $x = 46$, a sample value of the Gaussian $(61, 10)$ random variable. Express your test score as a sample value of the standard normal random variable, Z.

Equation (3.54) indicates that $z = (46 - 61)/10 = -1.5$. Therefore your score is 1.5 *standard deviations less than the expected value.*

Example 3.16 If X is the Gaussian $(61, 10)$ random variable, what is $P[X \leq 46]$?

Applying Theorem 3.14, Theorem 3.15 and the result of Example 3.15, we have

$$P[X \leq 46] = F_X(46) = \Phi(-1.5) = 1 - \Phi(1.5) = 1 - 0.933 = 0.067. \quad (3.55)$$

This suggests that if your test score is 1.5 standard deviations below the expected value, you are in the lowest 6.7% of the population of test takers.

Example 3.17 If X is a Gaussian random variable with $\mu = 61$ and $\sigma = 10$, what is $P[51 \leq X \leq 71]$?

Applying Equation (3.54), we find that the event $\{51 < X \leq 71\}$ corresponds to $\{-1 < Z \leq 1\}$. The probability of this event is

$$\Phi(1) - \Phi(-1) = \Phi(1) - [1 - \Phi(1)] = 2\Phi(1) - 1 = 0.683. \quad (3.56)$$

Definition 3.11 ***Standard Normal Complementary CDF***

*The **standard normal complementary CDF** is*

$$Q(z) = P[Z > z] = \frac{1}{\sqrt{2\pi}} \int_z^{\infty} e^{-u^2/2}\, du = 1 - \Phi(z).$$

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2}\, du. \quad (3.132)$$

It is related to the Gaussian CDF by

$$\Phi(x) = \frac{1}{2} + \frac{1}{2}\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right), \quad (3.133)$$

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Definition 4.17

Bivariate Gaussian Random Variables

Random variables X and Y have a ***bivariate Gaussian PDF*** *with parameters μ_1, σ_1, μ_2, σ_2, and ρ if*

$$f_{X,Y}(x,y) = \frac{\exp\left[-\dfrac{\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - \frac{2\rho(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \left(\frac{y-\mu_2}{\sigma_2}\right)^2}{2(1-\rho^2)}\right]}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}},$$

where μ_1 and μ_2 can be any real numbers, $\sigma_1 > 0$, $\sigma_2 > 0$, and $-1 < \rho < 1$.

Figure 4.5 illustrates the bivariate Gaussian PDF for $\mu_1 = \mu_2 = 0$, $\sigma_1 = \sigma_2 = 1$, and three values of ρ. When $\rho = 0$, the joint PDF has the circular symmetry of a sombrero. When $\rho = 0.9$, the joint PDF forms a ridge over the line $x = y$, and when $\rho = -0.9$ there is a ridge over the line $x = -y$. The ridge becomes increasingly steep as $\rho \to \pm 1$.

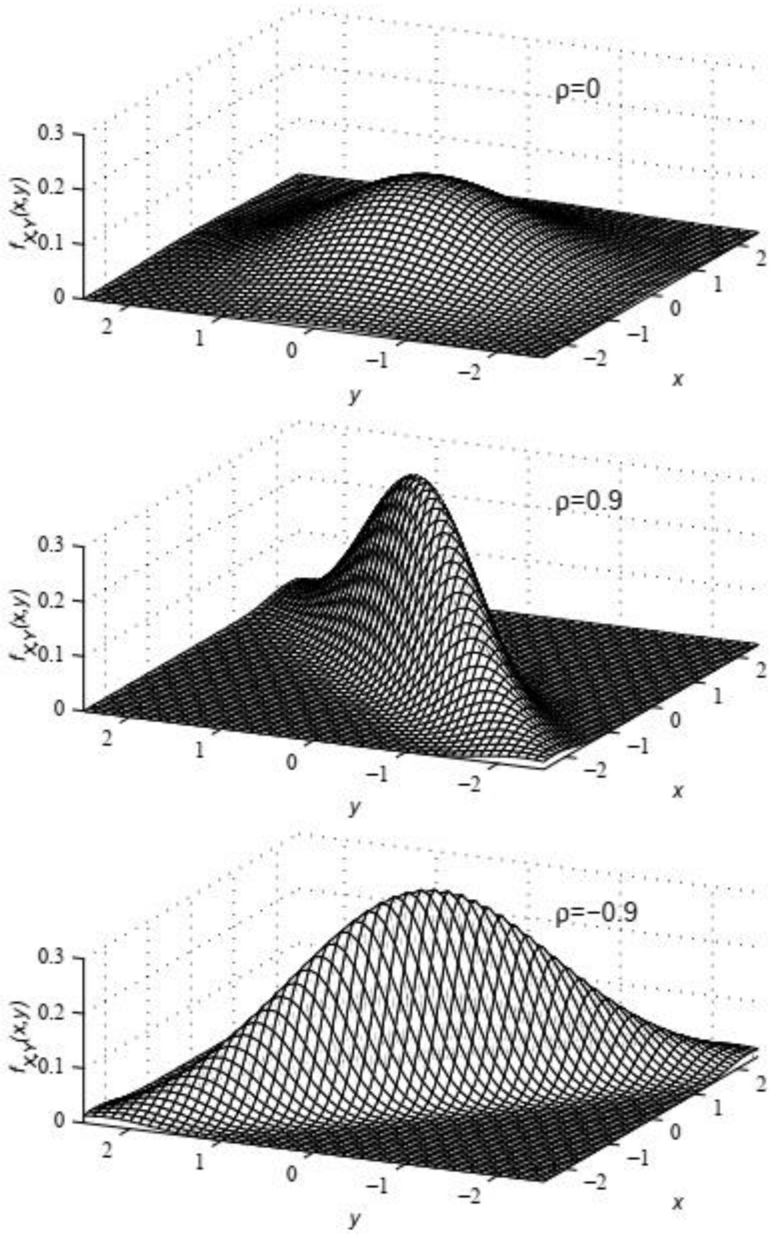

Figure 4.5 The Joint Gaussian PDF $f_{X,Y}(x, y)$ for $\mu_1 = \mu_2 = 0$, $\sigma_1 = \sigma_2 = 1$, and three values of ρ.

To examine mathematically the properties of the bivariate Gaussian PDF, we define

$$\tilde{\mu}_2(x) = \mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x - \mu_1), \qquad \tilde{\sigma}_2 = \sigma_2\sqrt{1 - \rho^2}, \tag{4.145}$$

$$\frac{1}{\sigma_1\sqrt{2\pi}\;\sigma_2\sqrt{1-\rho^2}\sqrt{2\pi}}$$

e: $-\frac{1}{2}\left\{\left(\frac{x-\mu_1}{\sigma_1}\right)^2 + \frac{\cancel{\sigma_2^2}\left[\left(\frac{y-\mu_2}{\sigma_2}\right) - \rho\left(\frac{x-\mu_1}{\sigma_1}\right)\right]^2}{\cancel{\sigma_2^2}(1-\rho^2)}\right\} =$

$$-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - \cancel{\rho^2\left(\frac{x-\mu_1}{\sigma_1}\right)^2} + \left[\left(\frac{y-\mu_2}{\sigma_2}\right)^2 + \cancel{\rho^2\left(\frac{x-\mu_1}{\sigma_1}\right)^2} - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right)\right]\right\}$$

$$\left[y-\tilde{\mu}_2(x)\right]^2 = \sigma_1^2\left(\frac{y-\mu_2}{\sigma_2} - \rho\frac{\frac{\sigma_2}{\sigma_1}(x-\mu_1)}{\sigma_2}\right)^2$$

and manipulate the formula in Definition 4.17 to obtain the following expression for the joint Gaussian PDF:

$$f_{X,Y}(x,y) = \frac{1}{\sigma_1\sqrt{2\pi}}e^{-(x-\mu_1)^2/2\sigma_1^2}\frac{1}{\tilde{\sigma}_2\sqrt{2\pi}}e^{-(y-\tilde{\mu}_2(x))^2/2\tilde{\sigma}_2^2}. \tag{4.146}$$

Equation (4.146) expresses $f_{X,Y}(x,y)$ as the product of two Gaussian PDFs, one with parameters μ_1 and σ_1 and the other with parameters $\tilde{\mu}_2$ and $\tilde{\sigma}_2$. This formula plays a key role in the proof of the following theorem.

Theorem 4.28

If X and Y are the bivariate Gaussian random variables in Definition 4.17, X is the Gaussian (μ_1,σ_1) random variable and Y is the Gaussian (μ_2,σ_2) random variable:

$$f_X(x) = \frac{1}{\sigma_1\sqrt{2\pi}}e^{-(x-\mu_1)^2/2\sigma_1^2} \qquad f_Y(y) = \frac{1}{\sigma_2\sqrt{2\pi}}e^{-(y-\mu_2)^2/2\sigma_2^2}.$$

Proof Integrating $f_{X,Y}(x,y)$ in Equation (4.146) over all y, we have

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy \tag{4.147}$$

$$= \frac{1}{\sigma_1\sqrt{2\pi}}e^{-(x-\mu_1)^2/2\sigma_1^2}\underbrace{\int_{-\infty}^{\infty}\frac{1}{\tilde{\sigma}_2\sqrt{2\pi}}e^{-(y-\tilde{\mu}_2(x))^2/2\tilde{\sigma}_2^2}\,dy}_{1} \tag{4.148}$$

The integral above the bracket equals 1 because it is the integral of a Gaussian PDF. The remainder of the formula is the PDF of the Gaussian (μ_1,σ_1) random variable. The same reasoning with the roles of X and Y reversed leads to the formula for $f_Y(y)$.

Method2:

$$f(y) = \int_{-\infty}^{+\infty} f(x,y)\,dx$$

Substituting

$$v = (x-\mu_x)/\sigma_x \quad , \quad dv = dx/\sigma_x$$

gives

$$f(y) = \int_{-\infty}^{+\infty} \frac{1}{2\pi\sigma_y\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2 - \frac{1}{2(1-\rho^2)}\left(v-\rho\frac{y-\mu_y}{\sigma_y}\right)^2\right] dv$$

Now substituting

$$w = \frac{v-\rho(y-\mu_y)/\sigma_y}{\sqrt{1-\rho^2}} \quad , \quad dw = \frac{dv}{\sqrt{1-\rho^2}}$$

gives

$$f(y) = \int_{-\infty}^{+\infty} \frac{1}{2\pi\sigma_y} \exp\left[-\frac{1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2 - \frac{1}{2}w^2\right] dw$$

and thus

$$f(y) = \frac{1}{2\pi\sigma_y} \exp\left[-\frac{1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2\right] \int_{-\infty}^{+\infty} \exp\left[-\frac{1}{2}w^2\right] dw$$

Since

$$\int_{-\infty}^{+\infty} \exp\left[-\frac{1}{2}w^2\right] dw = \sqrt{2\pi}$$

we obtain

$$f(y) = \frac{1}{\sqrt{2\pi}\sigma_y} \exp\left[-\frac{1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]$$

Conditional distributions

Theorem 4.29 is the result of dividing $f_{X,Y}(x,y)$ in Equation (4.146) by $f_X(x)$ to obtain $f_{Y|X}(y|x)$.

Theorem 4.29

If X and Y are the bivariate Gaussian random variables in Definition 4.17, the conditional PDF of Y given X is

$$f_{Y|X}(y|x) = \frac{1}{\tilde{\sigma}_2\sqrt{2\pi}} e^{-(y-\tilde{\mu}_2(x))^2/2\tilde{\sigma}_2^2},$$

where, given $X = x$, the conditional expected value and variance of Y are

$$\tilde{\mu}_2(x) = \mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x - \mu_1), \qquad \tilde{\sigma}_2^2 = \sigma_2^2(1 - \rho^2).$$

Similarly for f(x|y):

$$f(x|y) = \frac{f(x,y)}{f(y)}$$

Therefore

$$f(x|y) = \frac{\sqrt{2\pi}\sigma_y \exp\left[-\frac{1}{2(1-\rho^2)}\left(\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right)\right]}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}\exp\left[-\frac{1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]}$$

and thus

$$f(x|y) = \frac{1}{\sqrt{2\pi}\sigma_x\sqrt{1-\rho^2}}\exp\left[-\frac{1}{2\sigma_x^2(1-\rho^2)}\left[x - \left(\mu_x + \rho\frac{\sigma_x}{\sigma_y}(y-\mu_y)\right)\right]^2\right]$$

This density function thus corresponds to a univariate normal distribution with mean $\mu_x + \rho\frac{\sigma_x}{\sigma_y}(y - \mu_y)$ and variance $\sigma_x^2(1 - \rho^2)$.

The next theorem identifies ρ in Definition 4.17 as the correlation coefficient of X and Y, $\rho_{X,Y}$.

Theorem 4.31 *Bivariate Gaussian random variables X and Y in Definition 4.17 have correlation coefficient*

$$\rho_{X,Y} = \rho.$$

Definition 4.4 ***Covariance***
*The **covariance** of two random variables X and Y is*

$$\text{Cov}[X, Y] = E[(X - \mu_X)(Y - \mu_Y)].$$

Definition 4.8 ***Correlation Coefficient***
*The **correlation coefficient** of two random variables X and Y is*

$$\rho_{X,Y} = \frac{\text{Cov}[X, Y]}{\sqrt{\text{Var}[X]\,\text{Var}[Y]}} = \frac{\text{Cov}[X, Y]}{\sigma_X \sigma_Y}.$$

Theorem 4.12 *For random variables X and Y, the expected value of $W = g(X, Y)$ is*

$$\textit{Discrete:} \quad E[W] = \sum_{x \in S_X} \sum_{y \in S_Y} g(x, y) P_{X,Y}(x, y),$$

$$\textit{Continuous:} \; E[W] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{X,Y}(x, y)\, dx\, dy.$$

Proof Substituting μ_1, σ_1, μ_2, and σ_2 for μ_X, σ_X, μ_Y, and σ_Y in Definition 4.4 and Definition 4.8, we have

$$\rho_{X,Y} = \frac{E[(X - \mu_1)(Y - \mu_2)]}{\sigma_1 \sigma_2}. \tag{4.149}$$

To evaluate this expected value, we use the substitution $f_{X,Y}(x, y) = f_{Y|X}(y|x) f_X(x)$ in the double integral of Theorem 4.12. The result can be expressed as

$$\rho_{X,Y} = \frac{1}{\sigma_1 \sigma_2} \int_{-\infty}^{\infty} (x - \mu_1) \left(\int_{-\infty}^{\infty} (y - \mu_2) f_{Y|X}(y|x)\, dy \right) f_X(x)\, dx \tag{4.150}$$

$$= \frac{1}{\sigma_1 \sigma_2} \int_{-\infty}^{\infty} (x - \mu_1) E\left[Y - \mu_2 | X = x\right] f_X(x)\, dx \tag{4.151}$$

Because $E[Y|X = x] = \tilde{\mu}_2(x)$ in Theorem 4.29, it follows that

$$E\left[Y - \mu_2 | X = x\right] = \tilde{\mu}_2(x) - \mu_2 = \rho \frac{\sigma_2}{\sigma_1}(x - \mu_1) \tag{4.152}$$

Therefore,

$$\rho_{X,Y} = \frac{\rho}{\sigma_1^2} \int_{-\infty}^{\infty} (x - \mu_1)^2 f_X(x)\, dx = \rho, \tag{4.153}$$

because the integral in the final expression is $\mathrm{Var}[X] = \sigma_1^2$.

From Theorem 4.31, we observe that if X and Y are uncorrelated, then $\rho = 0$ and, from Theorems 4.29 and 4.30, $f_{Y|X}(y|x) = f_Y(y)$ and $f_{X|Y}(x|y) = f_X(x)$. Thus we have the following theorem.

Theorem 4.32 *Bivariate Gaussian random variables X and Y are uncorrelated if and only if they are independent.*

Theorem 4.31 identifies the parameter ρ in the bivariate gaussian PDF as the correlation coefficient $\rho_{X,Y}$ of bivariate Gaussian random variables X and Y. Theorem 4.17 states that for any pair of random variables, $|\rho_{X,Y}| < 1$, which explains the restriction $|\rho| < 1$ in Definition 4.17. Introducing this inequality to the formulas for conditional variance in Theorem 4.29 and Theorem 4.30 leads to the following inequalities:

$$\mathrm{Var}\left[Y|X = x\right] = \sigma_2^2(1 - \rho^2) \le \sigma_2^2, \tag{4.154}$$

$$\mathrm{Var}\left[X|Y = y\right] = \sigma_1^2(1 - \rho^2) \le \sigma_1^2. \tag{4.155}$$

These formulas state that for $\rho \neq 0$, learning the value of one of the random variables leads to a model of the other random variable with reduced variance. This suggests that learning the value of Y reduces our uncertainty regarding X.

Definition of the Bivariate Normal Distribution

Suppose that Z_1 and Z_2 are independent random variables, each of which has a standard normal distribution. Then the joint p.d.f. $g(z_1, z_2)$ of Z_1 and Z_2 is specified for all values of z_1 and z_2 by the equation

$$g(z_1, z_2) = \frac{1}{2\pi} \exp\left[-\frac{1}{2}(z_1^2 + z_2^2) \right]. \tag{5.12.1}$$

For constants μ_1, μ_2, σ_1, σ_2, and ρ such that $-\infty < \mu_i < \infty$ $(i = 1, 2)$, $\sigma_i > 0$ $(i = 1, 2)$, and $-1 < \rho < 1$,

we shall now define two new random variables X_1 and X_2 as follows:

$$X_1 = \sigma_1 Z_1 + \mu_1,$$
$$X_2 = \sigma_2 \left[\rho Z_1 + (1 - \rho^2)^{1/2} Z_2 \right] + \mu_2. \tag{5.12.2}$$

If the relations (5.12.2) are solved for Z_1 and Z_2 in terms of X_1 and X_2,

$$\left(\frac{X_1-\mu_1}{\sigma_1}\right)^2 = Z_1^2 \qquad \frac{X_2-\mu_2}{\sigma_2} - \rho Z_1 = (1-\rho^2)^{1/2} Z_2$$

$$\left(\frac{X_2-\mu_2}{\sigma_2} - \rho\left(\frac{X_1-\mu_1}{\sigma_1}\right)\right)^2 = \left((1-\rho^2)^{1/2} Z_2\right)^2 = (1-\rho^2) Z_2^2$$

$$Z_1^2 + Z_2^2 = \left(\frac{X_1-\mu_1}{\sigma_1}\right)^2 + \frac{\left(\frac{X_2-\mu_2}{\sigma_2} - \rho\left(\frac{X_1-\mu_1}{\sigma_1}\right)\right)^2}{(1-\rho^2)} = \frac{(1-\rho^2)\left(\frac{X_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{X_2-\mu_2}{\sigma_2} - \rho\left(\frac{X_1-\mu_1}{\sigma_1}\right)\right)^2}{(1-\rho^2)}$$

$$= \frac{(1-\rho^2)\left(\frac{X_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{X_2-\mu_2}{\sigma_2}\right)^2 + \rho^2\left(\frac{X_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{X_1-\mu_1}{\sigma_1}\right)\left(\frac{X_2-\mu_2}{\sigma_2}\right)}{(1-\rho^2)}$$

$$= \frac{\left(\frac{X_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{X_2-\mu_2}{\sigma_2}\right)^2 - 2\rho\left(\frac{X_1-\mu_1}{\sigma_1}\right)\left(\frac{X_2-\mu_2}{\sigma_2}\right)}{(1-\rho^2)}$$

The transformation from Z_1 and Z_2 to X_1 and X_2 is a linear transformation; and it will be found that the determinant Δ of the matrix of coefficients of Z_1 and Z_2 has the value $\Delta = (1-\rho^2)^{1/2}\sigma_1\sigma_2$. Therefore, as discussed in Section 3.9, the Jacobian J of the inverse transformation from X_1 and X_2 to Z_1 and Z_2 is

$$J = \frac{1}{\Delta} = \frac{1}{(1-\rho^2)^{1/2}\sigma_1\sigma_2}. \tag{5.12.3}$$

Since $J > 0$, the value of $|J|$ is equal to the value of J itself. If the relations (5.12.2) are solved for Z_1 and Z_2 in terms of X_1 and X_2, then the joint p.d.f. $f(x_1, x_2)$ can be obtained by replacing z_1 and z_2 in Eq. (5.12.1) by their expressions in terms of x_1 and x_2, and then multiplying by $|J|$. It can be shown that the result is, for $-\infty < x_1 < \infty$ and $-\infty < x_2 < \infty$,

$$f(x_1, x_2) = \frac{1}{2\pi(1-\rho^2)^{1/2}\sigma_1\sigma_2} \exp\left\{ -\frac{1}{2(1-\rho^2)}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]\right\}. \tag{5.12.4}$$

When the joint p.d.f. of two random variables X_1 and X_2 is of the form in Eq. (5.12.4), it is said that X_1 and X_2 have a *bivariate normal distribution*. The means and the variances of the bivariate normal distribution specified by Eq. (5.12.4) are easily derived from the definitions in Eq. (5.12.2). Because Z_1 and Z_2 are independent and each has mean 0 and

variance 1, it follows that $E(X_1) = \mu_1$, $E(X_2) = \mu_2$, $\mathrm{Var}(X_1) = \sigma_1^2$, and $\mathrm{Var}(X_2) = \sigma_2^2$. Furthermore, it can be shown by using Eq. (5.12.2) that $\mathrm{Cov}(X_1, X_2) = \rho\sigma_1\sigma_2$. Therefore, the correlation of X_1 and X_2 is simply ρ. In summary, if X_1 and X_2 have a bivariate normal distribution for which the p.d.f. is specified by Eq. (5.12.4), then

$$E(X_i) = \mu_i \quad \text{and} \quad \mathrm{Var}(X_i) = \sigma_i^2 \quad \text{for } i = 1, 2.$$

Also,

$$\rho(X_1, X_2) = \rho.$$

Reminder of pdf of 2 functions Garcia 3rd ed page 314 (329/833)

Let the random variables V and W be defined by two functions of X and Y:

$$V = g_1(X, Y) \qquad \text{and} \qquad W = g_2(X, Y). \tag{6.20}$$

Assume that the functions $v(x, y)$ and $w(x, y)$ are invertible in the sense that the equations $v = g_1(x, y)$ and $w = g_2(x, y)$ can be solved for x and y, that is,

$$x = h_1(v, w) \text{ and } y = h_2(v, w).$$

$$J(x, y) = \det\begin{bmatrix} \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} \end{bmatrix}.$$

The determinant $J(x, y)$ is called the **Jacobian** of the transformation. The Jacobian of the inverse transformation is given by

$$J(v, w) = \det\begin{bmatrix} \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \end{bmatrix}.$$

It can be shown that

$$|J(v, w)| = \frac{1}{|J(x, y)|}.$$

We therefore conclude that the joint pdf of V and W can be found using either of the following expressions:

$$f_{V,W}(v, w) = \frac{f_{X,Y}(h_1(v, w), (h_2(v, w))}{|J(x, y)|} \tag{6.22a}$$

$$= f_{X,Y}(h_1(v, w), (h_2(v, w))|J(v, w)|. \tag{6.22b}$$

Joint vector gaussian page 325 garcia

JOINTLY GAUSSIAN RANDOM VECTORS

The random variables $X_1, X_2, \ldots, X_n$ are said to be jointly Gaussian if their joint pdf is given by

$$f_{\mathbf{X}}(\mathbf{x}) \triangleq f_{X_1,X_2,\ldots,X_n}(x_1, \ldots, x_n) = \frac{\exp\{-\frac{1}{2}(\mathbf{x} - \mathbf{m})^{\mathrm{T}} K^{-1}(\mathbf{x} - \mathbf{m})\}}{(2\pi)^{n/2}|K|^{1/2}}, \tag{6.42a}$$

where $\mathbf{x}$ and $\mathbf{m}$ are column vectors defined by

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \mathbf{m} = \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_n \end{bmatrix} = \begin{bmatrix} E[X_1] \\ E[X_2] \\ \vdots \\ E[X_n] \end{bmatrix}$$

and K is the covariance matrix that is defined by

$$K = \begin{bmatrix} \mathrm{VAR}(X_1) & \mathrm{COV}(X_1, X_2) & \ldots & \mathrm{COV}(X_1, X_n) \\ \mathrm{COV}(X_2, X_1) & \mathrm{VAR}(X_2) & \ldots & \mathrm{COV}(X_2, X_n) \\ \vdots & \vdots & & \vdots \\ \mathrm{COV}(X_n, X_1) & \ldots & & \mathrm{VAR}(X_n) \end{bmatrix}. \tag{6.42b}$$

The $(.)^{\mathrm{T}}$ in Eq. (6.42a) denotes the transpose of a matrix or vector. Note that the covariance matrix is a symmetric matrix since $\mathrm{COV}(X_i, X_j) = \mathrm{COV}(X_j, X_i)$.

Equation (6.42a) shows that *the pdf of jointly Gaussian random variables is completely specified by the individual means and variances and the pairwise covariances*. It can be shown using the joint characteristic function that all the marginal pdf's associated with Eq. (6.42a) are also Gaussian and that these too are completely specified by the same set of means, variances, and covariances.

Example 6.20

Verify that the two-dimensional Gaussian pdf given in Eq. (5.61a) has the form of Eq. (6.42a).

The covariance matrix for the two-dimensional case is given by

$$K = \begin{bmatrix} \sigma_1^2 & \rho_{X,Y}\sigma_1\sigma_2 \\ \rho_{X,Y}\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix},$$

where we have used the fact the $\mathrm{COV}(X_1, X_2) = \rho_{X,Y}\sigma_1\sigma_2$. The determinant of K is $\sigma_1^2 \sigma_2^2(1 - \rho^2_{X,Y})$ so the denominator of the pdf has the correct form. The inverse of the covariance matrix is also a real symmetric matrix:

$$K^{-1} = \frac{1}{\sigma_1^2\sigma_2^2(1 - \rho_{X,Y}^2)}\begin{bmatrix} \sigma_2^2 & -\rho_{X,Y}\sigma_1\sigma_2 \\ -\rho_{X,Y}\sigma_1\sigma_2 & \sigma_1^2 \end{bmatrix}.$$

The term in the exponent is therefore

$$\frac{1}{\sigma_1^2\sigma_2^2(1 - \rho_{X,Y}^2)}(x - m_1, y - m_2)\begin{bmatrix} \sigma_2^2 & -\rho_{X,Y}\sigma_1\sigma_2 \\ -\rho_{X,Y}\sigma_1\sigma_2 & \sigma_1^2 \end{bmatrix}\begin{bmatrix} x - m_1 \\ y - m_2 \end{bmatrix}$$

$$= \frac{1}{\sigma_1^2\sigma_2^2(1 - \rho_{X,Y}^2)}(x - m_1, y - m_2)\begin{bmatrix} \sigma_2^2(x - m_1) - \rho_{X,Y}\sigma_1\sigma_2(y - m_2) \\ -\rho_{X,Y}\sigma_1\sigma_2(x - m_1) + \sigma_1^2(y - m_2) \end{bmatrix}$$

$$= \frac{((x - m_1)/\sigma_1)^2 - 2\rho_{X,Y}((x - m_1)/\sigma_1)((y - m_2)/\sigma_2) + ((y - m_2)/\sigma_2)^2}{(1 - \rho_{X,Y}^2)}.$$

Thus the two-dimensional pdf has the form of Eq. (6.42a).